

Formal asymptotic expansion of the Faddeev-Green function in unbounded domains

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Abstract

We consider the Faddeev-Green function in the three-dimensional space and in a slab, and we construct formal asymptotic expansions for the large complex parameter appearing in this function. The basic idea of the construction is to express the Faddeev-Green function through the standard exponential integral and to use the standard asymptotic expansion of this special function. In the three-dimensional space, the constructed expansion of the Faddeev-Green function clearly suggests the form of the rigorous estimate proved by Sylvester and Uhlmann. and which is the basis of complex-geometric optics' techniques in inverse problems. A similar estimate is suggested for the slab case .

1 Introduction

Complex plane waves of the form $\exp(i\zeta \cdot \mathbf{x})$, $\mathbf{x} \in \mathbb{R}^3$, with $\zeta \in \mathbb{C}^3$ satisfying $\zeta \cdot \zeta = t$, $\zeta \cdot \zeta = 0$, are solutions of the Laplace equation $\Delta \exp(i\zeta \cdot \mathbf{x}) = 0$ and they have two important properties. First, these solutions are exponential decaying/growing on each side of the surface $\mathbf{x} \cdot \zeta = 0$. Second, the span of the products of two complex plane waves is dense in $L^2(\Omega)$, $(\Omega) \subset \mathbb{R}^3$ being a bounded domain (this property is referred as the completeness of complex geometric optics (CGO) solutions). This property was first observed by Calderon [1], who used it to prove unique identifiability of conductivity from the linearised Dirichlet-to-Neumann map. Complex geometric optics solutions are extensions of complex plane waves to more general equations than the Laplace equations. The basic example of this extension is the case of Schrodinger operator done by Sylvester and Uhlmann. [2]. See also [3] for a nice exposition, [4] for a detailed presentation of rigorous results for CGO solutions for systems, based on the theory of Ψ DOs, and [5] for the use of CGOs in inverse spectral theory.

The construction of CGOs for an operator P , acting say on \mathbb{R}^3 , relies on the the Green function G_ζ of the Faddeev operator $P_\zeta = \exp(-i\zeta \cdot \mathbf{x})P\exp(i\zeta \cdot \mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^3$, $\zeta \in \mathbb{C}^3$. The asymptotic expansion of the Faddeev-Green function (abbreviated in the sequel as FG) G_ζ for large $s = |\zeta|$ is a key ingredient in the study of uniqueness of inverse boundary problems, un the construction of solutions to the inverse spectral problems, and even more in the numerical solution of tomographic problems [6]

In this note we consider the asymptotic expansion of the FG corresponding to the Laplacian $P = \Delta$ defined in \mathbb{R}^3 and to the Helmholtz operator $P = \Delta + k_0^2$, $k_0 > 0$ defined in the slab $\mathbb{R}_H^3 = \{(x, y, z) \mid (x, y) \in \mathbb{R}^2, 0 < z < H\}$. In the former case we "retrieve" the $1/s$ estimate which has been proven in [2]. In the later case we derive a same estimate with $s = |\zeta_{||}|$, where $\zeta_{||}$ is the component of ζ which is parallel to the flat boundaries of the slab.

2 The Faddeev-Green function in \mathbb{R}^3

Let $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$, and $\zeta = (k, \ell, m) \in \mathbb{C}^3$, such that $\zeta \cdot \zeta = k_0^2$. $k_0 > 0$. We consider the Green function $G(\mathbf{x})$ for the Helmholtz equation, which satisfies the equation

$$(\Delta + k_0^2) G(\mathbf{x}) = \delta(\mathbf{x}), \quad k_0 > 0, \quad (2.1)$$

and we introduce the decomposition

$$G(\mathbf{x}) = \exp(i\zeta \cdot \mathbf{x}) G_\zeta(\mathbf{x}). \quad (2.2)$$

Then $G_\zeta(\mathbf{x})$ satisfies the equation

$$\Delta_\zeta G_\zeta(\mathbf{x}) = \delta(x)\delta(y)\delta(z) \quad (x, y, z) \in \mathbb{R}^3, \quad (2.3) \quad \{\mathbf{fgfp}\}$$

where

$$\Delta_\zeta = \Delta + 2i\zeta \cdot \nabla, \quad (2.4)$$

is the Faddeev Laplacian, and we call G_ζ the Faddeev-Green function (abbreviated in the sequel as FG).

The function G_ζ is constructed by Fourier transform, and it is given by

$$G_\zeta(\mathbf{x}) = \int_{\mathbb{R}^3} \exp(i\xi \mathbf{x}) (\xi^2 + 2\zeta \cdot \xi)^{-1} d\xi. \quad (2.5)$$

We introduce the spherical coordinates

$$\begin{aligned} x &= R \sin \psi \cos \omega, & y &= R \sin \psi \sin \omega, \\ z &= R \cos \psi, & 0 \leq \psi \leq \pi, & \quad 0 \leq \omega \leq 2\pi \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} \xi_1 &= r \sin \theta \cos \phi, & \xi_2 &= r \sin \theta \sin \phi, \\ \xi_3 &= r \cos \theta, & 0 \leq \theta \leq \pi, & \quad 0 \leq \phi \leq 2\pi \end{aligned} \quad (2.7)$$

in the physical and the Fourier space, respectively.

We define $\dot{\zeta} = (\dot{k}, \dot{\ell}, \dot{m})$, such that $\zeta = s\dot{\zeta}$, where $s = |\zeta|$, and we have

$$\zeta \cdot \xi = \alpha(\theta, \phi) rs, \quad \alpha = \alpha(\theta, \phi) = (\dot{k} \cos \phi + \dot{\ell} \sin \phi) \sin \theta + \dot{m} \cos \theta,$$

and

$$\xi \cdot \mathbf{x} = \beta(\theta, \phi; \psi, \omega) r R, \quad \beta = \beta(\theta, \phi; \psi, \omega) = \sin \theta \sin \psi \cos(\phi - \omega) + \cos \theta \cos \psi$$

By the change of coordinates from Cartesian to spherical ones, we also have $d\xi = r^2 \sin \theta dr d\theta d\phi$.

Then, we rewrite G_ζ in the form

$$G_\zeta(\mathbf{x}) = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \int_0^\infty \frac{r \exp(iR\beta r)}{r + 2s\alpha} dr. \quad (2.8) \quad \{\mathbf{pgz}\}$$

We now observe that we can express the inner radial integral

$$I(R, s; \alpha, \beta) = \int_0^\infty \frac{r \exp(iR\beta r)}{r + 2s\alpha} dr \quad (2.9) \quad \{\mathbf{rint}\}$$

in terms of the exponential integral ([8], Ch. 3)

$$Ei(-z) = - \int_z^\infty \frac{\exp(-u)}{u} du, \quad z \in \mathbb{C}, \quad |\arg(z)| < \pi. \quad (2.10) \quad \{\mathbf{ei}\}$$

In fact, by defining $A = 2s\alpha$, $B = -iR\beta$. we obviously have $|\arg(AB)| < \pi$, and therefore

$$I(R, s; \alpha, \beta) = A \left(\exp(AB) Ei(-AB) + \frac{1}{AB} \right) \quad (2.11)$$

Then, we use the asymptotic expansion of Ei ,

$$\exp(-z)Ei(z) = \sum_{k=0}^n \frac{k!}{z^{k+1}} + O(|z|^{-n-2}) , \quad |z| \rightarrow \infty , \quad (2.12) \quad \{\text{asei}\}$$

$$|\arg(-z)| \leq \pi - \delta , \quad \delta > 0 , \quad \forall n \in \mathbb{N}$$

and we get the following asymptotic expansion of the radial integral (2.9), for large s ,

$$I(R, s; \alpha, \beta) = \sum_{k=1}^n \frac{k!}{i^{k+1} 2^k \alpha^k} \frac{1}{R^{k+1} \beta^{k+1}} \frac{1}{s^k} + O(s^{-n-1}) \quad (2.13) \quad \{\text{iexp}\}$$

$$s \rightarrow \infty , \quad -\pi/2 < \arg \alpha < 3\pi/2 .$$

Now, we substitute (2.13) into (2.8) and we *formally* integrate term by term the expansion of I . This leads to the formal expansion

$$G_\zeta(\mathbf{x}) = \sum_{k=1}^n \frac{k!}{i^{k+1} 2^k} \left(\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \frac{1}{\alpha^k(\theta, \phi)} \frac{1}{\beta^{k+1}(\theta, \phi; \psi, \omega)} \right) \frac{1}{R^{k+1}} \frac{1}{s^k} + \quad (2.14) \quad \{\text{fas}\}$$

$$+ O(s^{-(n+1)}) , \quad s \rightarrow \infty , \quad n = 1, 2, \dots$$

The angle integrals appearing in the coefficients of the expansion for $k \geq 2$ are distributions although for $k = 1$ is the coefficient is a smooth function. The first term of (2.14) leads for large $s = |\zeta|$, after averaging the dependence on the angle coordinates of the spherical system by taking appropriate norms, to the $\frac{1}{s}$ estimate for the convolution operator ($G_\zeta *$) which has been proved rigorously by Sylvester and Uhlmann [2].

3 The Faddeev-Green function in the slab

We now consider the Green function for the Helmholtz equation in the slab

$$\mathbb{R}_H^3 = \{\mathbf{x} = (x, y, z) \mid x, y \in \mathbb{R} , 0 < z < H\} ,$$

which satisfies the following boundary value problem

$$\begin{cases} (\Delta + k_0^2) G(\mathbf{x}) = \delta(x - x_0) \delta(y - y_0) \delta(z - z_0) & (x, y, z) \in \mathbb{R}_H^3 \\ G_\zeta(x, y, z = 0) = 0 , \\ \partial_z G(x, y, z = H) = 0 . \end{cases} \quad (3.1)$$

This problem typically arises in ocean acoustics and geophysics.

The corresponding FG satisfies now the boundary value problem

$$\begin{cases} \Delta_\zeta G_\zeta(\mathbf{x}) = \delta(x - x_0) \delta(y - y_0) \delta(z - z_0) & (x, y, z) \in \mathbb{R}^3 \\ G_\zeta(x, y, z = 0) = 0 , \\ (\partial_z + im) G_\zeta(x, y, z = H) = 0 , \end{cases} \quad (3.2) \quad \{\text{fgsp}\}$$

This FG has been introduced in [7] for studying the global uniqueness of an inverse boundary value problem in ocean acoustics, by extending the method of Sylvester and Uhlmann to this particular geometry. Here we extend the formal asymptotic procedure proposed in Section 2 , in order to derive a similar asymptotic expansion for the FG in the slab.

The problem (3.2), compared with (2.3), has the additional difficulty that it contains the component m of the complex spectral parameter in the mixed boundary condition at $z = H$,

We construct the solution of (3.2) by separation of variables. A long and cumbersome calculation leads to the eigenfunction series expansion

$$G_\zeta(\mathbf{x}) = -\frac{1}{2\pi^2 H} \exp(imz_0) \sum_{\nu \in \mathbb{Z}} \sin\left(\left(\nu + \frac{1}{2}\right) \frac{\pi z_0}{H}\right) \times \sin\left(\left(\nu + \frac{1}{2}\right) \frac{\pi z}{H}\right) I_\nu(x, y; x_0, y_0; k, \ell, m) \quad (3.3)$$

where the integrals

$$I_\nu(x, y; x_0, y_0; k, \ell, m) = \int \int_{\mathbb{R}^2} \frac{\exp(-i\xi(x - x_0) + \eta(y - y_0))}{\xi^2 + \eta^2 - 2(k\xi + \ell\eta) - \lambda_\nu^2} d\xi d\eta . \quad (3.4)$$

take care of the horizontal variation of FG, and

$$\lambda_\nu^2 = m^2 + (\nu + 1/2)^2 (\pi/H)^2 \quad (3.5)$$

are the eigenvalues of the separation spectral problem.

Note that the integrals I_ν can be expressed as single integrals in terms of Hankel functions, but we do not use this transformation here, since we want to express them in terms of the exponential integral and to proceed similarly to the full space case as in Section 2.

By rotation of the horizontal components of the complex parameter $\zeta = (\zeta_{||}, m)$ we can choose $\zeta_{||} = (k, \ell) = (s, i s \dot{\ell}_I)$. In the sequel the large parameter is the length $s = |\zeta_{||}|$ and it is related only to the horizontal variation of FG.

We introduce polar coordinates in the horizontal physical and Fourier plane

$$\begin{aligned} x - x_0 &= R \cos \theta , & y - y_0 &= R \sin \theta , & 0 \leq \theta < 2\pi \\ \xi &= r \cos \phi , & \eta &= r \sin \phi , & 0 \leq \phi < 2\pi \end{aligned} \quad (3.6)$$

and we rewrite the integrals I_ν in the form

$$I_\nu(x, y; x_0, y_0; k, \ell, m) = \int_0^{2\pi} d\phi M_\nu(\phi; R, \theta : k, \ell, m) \quad (3.7) \quad \{\text{in}\}$$

where

$$M_\nu(\phi; R, \theta : k, \ell, m) = \int_0^\infty \frac{\exp(-iR \cos(\theta - \phi)r)}{r^2 - 2(\cos \phi + i \dot{\ell} \sin \phi)sr - \lambda_\nu^2} dr . \quad (3.8) \quad \{\text{mn}\}$$

Then, we can express M_ν in terms of the exponential integral (2.10) as follows

$$\begin{aligned} M_\nu(\phi; R, \theta : k, \ell, m) &= \frac{1}{2\beta} (\alpha + \beta_\nu) \exp(-i\{\alpha R \cos(\theta - \phi)\}s) \\ &\times \left(\exp(is\beta_\nu R \cos(\theta - \phi)) Ei(is(\alpha - \beta_\nu)R \cos(\theta - \phi)) \right. \\ &\left. - \exp(-is\beta_\nu R \cos(\theta - \phi)) Ei(is(\alpha + \beta_\nu)R \cos(\theta - \phi)) \right) . \end{aligned} \quad (3.9)$$

The quantities α, β and ρ_ν are given by

$$\begin{aligned} \alpha &= \alpha(s) = s(\cos \phi + i \dot{\ell}_I \sin \phi) , \\ \beta_\nu^2 &= \beta_\nu^2(s) = \alpha^2 - (1 - \dot{\ell}_I) + \frac{\rho_\nu^2}{s^2} , \\ \rho_\nu^2 &= k_0^2 + (\nu + 1/2)^2 (\pi/H)^2 , \end{aligned} \quad (3.10)$$

and it holds that $|\arg(i(\alpha \pm \beta_\nu))| < \pi$, in particular $|\arg(\alpha \pm \beta_0)| < \pi/2$.

The asymptotic expansion of M_ν for large s is constructed by using (2.12) and the estimates

$$\beta_\nu(s) \sim \beta_0 + O(1/s^2) , \quad \beta_0 = \sqrt{\alpha^2 - (1 - \dot{\ell}_I)} , \quad s \rightarrow \infty . \quad (3.11)$$

The principal term is given by

$$M_\nu \sim -\frac{i}{R} \frac{\alpha + \sqrt{\alpha^2 - (1 - \dot{\ell}_I)}}{1 - \dot{\ell}_I^2} \frac{1}{\cos(\theta - \phi)} \frac{1}{s}, \quad s \rightarrow \infty. \quad (3.12) \quad \{\text{asmn}\}$$

Now we introduce (3.12) into (3.7), and we integrate analytically certain integrals with respect to the polar angle ϕ . In two steps. First, we first obtain

$$I_\nu \sim -\frac{i}{1 - \dot{\ell}_I^2} \frac{1}{R} \times \left(\int_0^{2\pi} \frac{\cos \phi + i \dot{\ell}_I \sin \phi}{\cos(\theta - \phi)} d\phi + \int_0^{2\pi} \frac{(\cos \phi + i \dot{\ell}_I \sin \phi)^2 - (1 - \dot{\ell}_I)^2}{\cos(\theta - \phi)} d\phi \right) \times \frac{1}{s} + \quad (3.13)$$

$$+ O_\nu(1/s^2),$$

and then

$$I_\nu \sim -\frac{i}{1 - \dot{\ell}_I^2} \frac{\exp(i\theta)}{R} \times \left(2\pi - \theta - i \left(\log(|\cos \theta|) - i\pi \chi_{(\pi/2 < \theta < 3\pi/2)} \right) \right) \times \frac{1}{s} + \quad (3.14)$$

$$+ O_\nu(1/s^2), \quad \nu = 1, 2, \dots$$

Finally, by substituting the expansions (3.14) into the FG (3.3), we derive the *formal* asymptotic expansion

$$G_\zeta(\mathbf{x}) \sim \frac{1}{2\pi^2} f(z; z_0) g(R, \theta; \dot{\ell}) \frac{\exp(imz_0 s)}{s} \times \left(1 + O(1/s^2) \right) \quad (3.15)$$

where

$$\begin{aligned} f(z; z_0) &= \cos \left(\frac{\pi(z + z_0)}{2H} \right) \sum_{\nu \in \mathbb{Z}} \delta(z + (z_0 - 2\nu H)) \\ &- \cos \left(\frac{\pi(z - z_0)}{2H} \right) \sum_{\nu \in \mathbb{Z}} \delta(z - (z_0 + 2\nu H)). \end{aligned} \quad (3.16)$$

and

$$g(R, \theta; \dot{\ell}_I) = -\frac{i}{1 - \dot{\ell}_I^2} \frac{\exp(i\theta)}{R} \left(2\pi - \theta - i \left(\log(|\cos \theta|) - i\pi \chi_{(\pi/2 < \theta < 3\pi/2)} \right) \right), \quad (3.17)$$

$\chi_{(a,b)}$ being the characteristic function of the interval (a, b) .

We observe that the principal term obeys again the $\frac{1}{s}$ dependence on the length s , which is however different that the corresponding parameter in the fill space case. We also observe that $f(z; z_0)$ has the form of a multiple scattering series, which is typically anticipated in slab type problems.

Given that eigenfunction series can be converted, in general, to multiple scattering expansions in terms of spherical waves (see the related analysis for (3.1) in [9]), it is an interesting open question whether or not the FG (3.3) can be expressed in terms of complex spherical waves like those used by Salo & Wang [10] for handling the inverse electro-conductivity problem in a slab (see also Ikehata's work [11] on the same problem, who used a generalisation of FG in infinite space to handle the same problem).

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